

Homology of graded Hecke algebras

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Abstract.

Let \mathbb{H} be a graded Hecke algebra with complex deformation parameters and Weyl group W . We show that the Hochschild, cyclic and periodic cyclic homologies of \mathbb{H} are all independent of the parameters, and compute them explicitly. We use this to prove that, if the deformation parameters are real, the collection of irreducible tempered \mathbb{H} -modules with real central character forms a \mathbb{Q} -basis of the representation ring of W .

Our method involves a new interpretation of the periodic cyclic homology of finite type algebras, in terms of the cohomology of a sheaf over the underlying complex affine variety.

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Introduction

Let \mathfrak{t}^* be a complex vector space containing a root system R with Weyl group W . Then W acts on the symmetric algebra $S(\mathfrak{t}^*)$ of \mathfrak{t}^* , so one can construct the crossed product algebra $W \ltimes S(\mathfrak{t}^*)$. Graded Hecke algebras are deformations of $W \ltimes S(\mathfrak{t}^*)$, depending on several parameters $k_\alpha \in \mathbb{C}$.

Lusztig [Lus2, Lus3] showed that graded Hecke algebras play an important role in the representation theory of affine Hecke algebras and of simple p -adic groups. For this reason it pays to understand the representations of graded Hecke algebras, and in particular to determine their irreducible representations.

We will approach this goal via the noncommutative geometry of these algebras. Every graded Hecke algebra \mathbb{H} is endowed with a natural filtration, whose associated graded algebra is $W \ltimes S(\mathfrak{t}^*)$. This gives rise to spectral sequences converging to various homologies of \mathbb{H} , and allows us to prove that:

Theorem 0.1. *(See Theorems 3.3 and 3.4.)*

The Hochschild and (periodic) cyclic homologies of \mathbb{H} and $W \ltimes S(\mathfrak{t}^)$ are isomorphic:*

$$\begin{aligned} HH_n(\mathbb{H}) &\cong HH_n(W \ltimes S(\mathfrak{t}^*)), \\ HC_n(\mathbb{H}) &\cong HC_n(W \ltimes S(\mathfrak{t}^*)), \\ HP_n(\mathbb{H}) &\cong HP_n(W \ltimes S(\mathfrak{t}^*)). \end{aligned}$$

Moreover the inclusion $\mathbb{C}[W] \rightarrow \mathbb{H}$ induces an isomorphism on HP_* .

However, the representation theoretic content of this result is not so obvious, for instance because the bijection on HH_* is not an isomorphism of $S(\mathfrak{t}^*)^W$ -modules. To clarify this we introduce a new interpretation of the periodic cyclic homology of an algebra. It is not specific for graded Hecke algebras, in fact it applies to all finite type algebras. Let X be a complex affine algebraic variety. It is known [KNS] that

$$HP_n(\mathcal{O}(X)) \cong \bigoplus_{m \in \mathbb{Z}} H^{n+2m}(X^{an}; \mathbb{C}), \quad (1)$$

where X^{an} is X with the analytic topology. Now let A be a finite type $\mathcal{O}(X)$ -algebra. To it we associate a sheaf S_A over X^{an} , which encodes the space of irreducible A -modules. We will establish an isomorphism

$$HP_n(A) \cong \bigoplus_{m \in \mathbb{Z}} \check{H}^{n+2m}(X^{an}; S_A), \quad (2)$$

where \check{H} stands for the Čech cohomology of a sheaf. The main advantage of (2) is that the extensive machinery of sheaf cohomology becomes available for the calculation of periodic cyclic homology.

For graded Hecke algebras we can really put (2) into effect, provided that the root system R is crystallographic and the parameters k_α are real. Probably these assumptions are not necessary for the final results, but we do use them in our proofs.

The representations of such graded Hecke algebras naturally come in series. Every such series is induced from a discrete series representation of a parabolic subalgebra and parametrized by a subspace of \mathfrak{t} . The individual representations in such a series can be equivalent, and they may be reducible. All this is described by the theory of intertwining operators, which the author discussed in the prequel [Sol3] to the

present paper. The upshot is that these series partition the primitive ideal spectrum of \mathbb{H} in disjoint subsets, and that every single such subset is in first approximation a vector space modulo a finite group.

This relatively simple shape allows us to compute the cohomology of the sheaf $S_{\mathbb{H}}$ from (2). As could be expected, there is no cohomology in strictly positive degrees. In degree zero we find that $\check{H}^0(\mathfrak{t}/W; S_{\mathbb{H}})$ is naturally isomorphic to the complex vector space whose basis is formed by the irreducible tempered \mathbb{H} -modules with real central character (these notions are defined in Section 5).

Thus we have two different ways to determine $HP_*(\mathbb{H})$:

- via a spectral sequence coming from the filtration of \mathbb{H} by degree,
- via the cohomology of a sheaf encoding the primitive ideal spectrum of \mathbb{H} .

The first method relates $HP_0(\mathbb{H})$ to $HP_0(\mathbb{C}[W])$, and hence to the irreducible W -representations. The second strategy results in a comparison between $HP_0(\mathbb{H})$ and the collection $\text{Irr}_0(\mathbb{H})$ of irreducible tempered \mathbb{H} -modules with real central character (modulo equivalence). Combining both strategies, we arrive at the main result of the paper:

Theorem 0.2. *(See Theorem 6.5.c.)*

Let R be a crystallographic root system with Weyl group W and let k_α ($\alpha \in R$) be real parameters. The representations $\{V|_W : V \in \text{Irr}_0(\mathbb{H})\}$ form a \mathbb{Q} -basis of the representation ring $R(W) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Hitherto this result was only known for “geometric” graded Hecke algebras, for which Ciubotaru [Ciu, Corollary 3.6] derived it from deep results of Lusztig. Here “geometric” means that these are the graded Hecke algebras that arise as the equivariant homology of specific algebraic varieties, as in [Lus2].

Moreover in these cases the expression of $\{V|_W : V \in \text{Irr}_0(\mathbb{H})\}$ with respect to the natural basis of $R(W)$ (suitably ordered) is a unitriangular matrix with integral coefficients. As Ciubotaru remarks, unitriangularity gives rise to a generalized Springer correspondence, via [Lus1]. Therefore it would be quite interesting to know whether this unitriangularity remains valid in our more general setting. Unfortunately the methods in this paper are unsuitable to detect it, since the complex coefficients of periodic cyclic homology destroy the torsion information.

Theorem 0.2 can be used to study several problems:

- determination of the unitary dual of graded Hecke algebras. Indeed, Theorem 0.2 (for geometric Hecke algebras) plays an important role in [Ciu].

Lusztig’s reduction theorems [Lus3] provide a concrete connection between representations of \mathbb{H} and of related affine Hecke algebras. This opens the road to

- the analogue of Theorem 0.1 for affine Hecke algebras.

Such an analogue would be useful to classify all irreducible representations of affine Hecke algebras, especially those with unequal parameters. Pursuing this line of thought even further, to the smooth representation theory of p -adic groups, could provide

- a promising tool to attack many cases of a conjecture of Aubert, Baum and Plymen [ABP1, ABP2], which describes Bernstein components in the smooth dual of a reductive p -adic group.

The author intends to study the last two applications in a forthcoming paper.

We conclude this introduction with a description of the various sections. The first Section recalls some general results on crossed products of commutative algebras with finite groups, in particular their representation theory and their homology. We state the definitions of graded Hecke algebras in Section 2. As Lusztig's reduction theorems involve groups of diagram automorphisms of R , we incorporate such groups in our algebras right from the start. This changes the notations somewhat, but does not make the proofs really more complicated. Section 3 is dedicated to proving Theorem 0.1. In the fourth Section we show how to relate the periodic cyclic homology of a finite type algebra to the cohomology of a sheaf, as in (2). Moreover we provide some examples of such sheaves. In Section 5 we recall the relevant results from [Sol3], which deal mostly with parabolically induced representations of graded Hecke algebras. In the final Section 6 we compute the cohomology of the sheaf $S_{\mathbb{H}}$, which leads to Theorem 0.2.

1 Crossed product algebras

We start with a discussion of a rather general kind of algebras, obtained from an action of a finite group on a commutative algebra. A lot is known about such crossed product algebras, and here we recall the results that we will use later on.

Let G be any group and A be an algebra over a field \mathbb{F} . Let $\beta : G \rightarrow \text{Aut}(A)$ be an action of G on A by algebra automorphisms. We endow the vector space $A \otimes_{\mathbb{F}} \mathbb{F}G$ with the multiplication

$$(a \otimes g) \cdot (a' \otimes g') = a \beta_g(a') \otimes gg' \quad a, a' \in A, g, g' \in G. \quad (3)$$

This defines an associative \mathbb{F} -algebra, denoted $A \rtimes G$ or $G \ltimes A$ and called the crossed product of A and G .

We specialize this in the direction of the algebras that are of most interest to us. Let X be a topological space and suppose that A is a subalgebra of $C(X; \mathbb{C})$ whose maximal ideal spectrum is precisely X . Furthermore we suppose that G is finite and acts on X by homeomorphisms, such that the induced action on $C(X; \mathbb{C})$ preserves A . Let \mathbb{C}_x be the onedimensional A -module with character $x \in X$ and write

$$G_x := \{g \in G : g(x) = x\},$$

$$I_x := \text{Ind}_A^{A \rtimes G} \mathbb{C}_x.$$

The representation theory of such algebras is not complicated, and can be obtained from classical results that go back to Frobenius and Clifford, see [Cli] or [CuRe, Theorem 11.1].

Theorem 1.1. a) $I_x \cong I_{x'}$ if and only if $Gx = Gx'$.

- b) $I_x \cong \text{Ind}_{A \rtimes G_x}^{A \rtimes G} (\mathbb{C}[G_x])$, where A acts on $\mathbb{C}[G_x]$ through the evaluation at x .
- c) I_x is completely reducible.
- d) Every irreducible $A \rtimes G$ -module is a direct summand of some I_x .
- e) The number of inequivalent irreducible constituents of I_x equals the number of conjugacy classes in G_x .

Proof. a) As an A -module

$$I_x = \bigoplus_{g \in G} \mathbb{C}g = \bigoplus_{g \in G} \mathbb{C}v_{gx},$$

and G acts by $g' \cdot v_{gx} = v_{g'gx}$. Similarly $I_{x'} = \bigoplus_{g \in G} \mathbb{C}\tilde{v}_{gx'}$. If $Gx = Gx'$ then

$$I_x \rightarrow I_{x'} : v_{gx} \mapsto \tilde{v}_{gx}$$

is an isomorphism of $A \rtimes G$ -modules. On the other hand, if $Gx \neq Gx'$, then $I_{x'}$ contains no A -weight spaces with weight x , so $I_{x'} \not\cong I_x$.

b)

$$\text{Ind}_{A \rtimes G_x}^{A \rtimes G} (\mathbb{C}[G_x]) = (A \rtimes G) \otimes_{A \rtimes G_x} \left(\bigoplus_{g \in G_x} \mathbb{C}v_{gx} \right) = \bigoplus_{g \in G} \mathbb{C}v_{gx} = I_x.$$

c) The group G permutes the A -weight spaces of I_x transitively, and the x -weight space is just $\mathbb{C}[G_x]$. Hence the functor $\text{Ind}_{A \rtimes G_x}^{A \rtimes G}$ provides a bijection between G_x -subrepresentations of $\mathbb{C}[G_x]$ and $A \rtimes G$ -submodules of I_x . Since $\mathbb{C}[G_x]$ is completely reducible, so is I_x .

d) Let (π, V) be an irreducible $A \rtimes G$ -module. There exists an x for which V contains a nonzero A -weight vector v_x . Then

$$\begin{aligned} \phi : I_x &\rightarrow V, \\ \phi((a \otimes g)v_x) &= \pi(a \otimes g)v_x \end{aligned}$$

is a surjective homomorphism of $A \rtimes G$ -modules and hence $V \cong I_x / \ker \phi$ is a quotient of I_x . But I_x is completely reducible, so V is also isomorphic to a direct summand of I_x .

e) The map $\text{Ind}_{A \rtimes G_x}^{A \rtimes G}$ above remains bijective on the level of isomorphism classes of irreducible modules. As is well known, the number of inequivalent irreducible constituents of the left regular representation $\mathbb{C}[G_x]$ equals the number of conjugacy classes in G_x . \square

Extended quotients are very useful to describe the homology of crossed products, so us recall these. Write

$$\tilde{X} = \{(g, x) \in G \times X : g(x) = x\}. \quad (4)$$

Then G acts on \tilde{X} by $g(g', x) = (gg'g^{-1}, g(x))$, and the extended quotient of X by G is defined as \tilde{X}/G . Write $X^g = \{x \in X : g(x) = x\}$, let $Z_G(g)$ be the centralizer

of g in G and let G/\sim be the collection of conjugacy classes in G . We can also construct the extended quotient as a disjoint union:

$$\begin{aligned}\tilde{X}/G &= \left(\bigcup_{g \in G} (g, X^g) \right) / G = \bigcup_{c \in G/\sim} \left(\bigcup_{g \in c} (g, X^g) \right) / G \\ &\cong \bigcup_{g/\sim \in G/\sim} (g, X^g / Z_G(g)) \cong \bigsqcup_{g/\sim \in G/\sim} X^g / Z_G(g).\end{aligned}\tag{5}$$

Suppose now that X is a nonsingular affine complex variety, and that G acts on it by algebraic isomorphisms. Then X^g is an algebraic variety as well, and with the slice theorem one deduces that X^g is a smooth manifold. Hence X^g and \tilde{X} are nonsingular affine varieties, although usually the latter has components of different dimensions.

For A we take the algebra $\mathcal{O}(X)$ of regular functions on X . Let $\Omega^n(X)$ denote the space of algebraic n -forms on X , and $H_{DR}^n(X)$ the (algebraic) De Rham-cohomology of X . The following result generalizes the classical Hochschild–Kostant–Rosenberg theorem:

Theorem 1.2. *Let X and G be as above. The Hochschild, cyclic and periodic cyclic homologies of $\mathcal{O}(X) \rtimes G$ are given by*

$$\begin{aligned}HH_n(\mathcal{O}(X) \rtimes G) &= \Omega^n(\tilde{X})^G, \\ HC_n(\mathcal{O}(X) \rtimes G) &= (\Omega^n(\tilde{X}) / d\Omega^{n-1}(\tilde{X}) \oplus H_{DR}^{n-2}(\tilde{X}) \oplus H_{DR}^{n-4}(\tilde{X}) \oplus \dots)^G, \\ HP_n(\mathcal{O}(X) \rtimes G) &= \left(\bigoplus_{m \in \mathbb{Z}} H_{DR}^{n+2m}(\tilde{X}) \right)^G.\end{aligned}$$

Proof. This result and various generalizations were proven by Brylinski and Nistor, see [Bry] and [Nis, Theorem 2.11]. \square

To make the above isomorphisms more explicit we need to recall the definitions of Hochschild and cyclic homology. For any unital algebra A and any A -bimodule M there is a differential complex $(C_*(A, M), b)$, where $C_n(A, M) = M \otimes A^{\otimes n}$ and $b : C_n(A, M) \rightarrow C_{n-1}(A, M)$ is defined on elementary tensors by

$$b(m \otimes a_1 \otimes \dots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + m a_1 \otimes a_1 \otimes \dots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \dots \otimes a_{n-1}.\tag{6}$$

This is known as the Hochschild complex, and its homology is denoted by $H_*(A, M)$. When $M = A$, we omit it from the notation and get $H_n(C_*(A), b) = HH_n(A)$.

We can extend $C_*(A)$ to a so-called mixed complex $\mathcal{B}_*(A)$:

$$\begin{array}{ccccc}\downarrow & & \downarrow & & \downarrow \\ A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\ \downarrow b & & \downarrow b & & \\ A^{\otimes 2} & \xleftarrow{B} & A & & \\ \downarrow b & & & & \\ A & & & & \end{array}\tag{7}$$

The degree on this complex is the sum of the horizontal and vertical positions. The operator $B : A^{\otimes(n+1)} \rightarrow A^{\otimes(n+2)}$ satisfies $bB + Bb = 0$ and is given explicitly by

$$B(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^n (-1)^{ni} (1 \otimes a_i - a_i \otimes 1) \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1}. \quad (8)$$

The cyclic homology of A is $HC_n(A) = H_n(\mathcal{B}_*(A), b, B)$.

The remainder of this section, which will only be used in Theorem 3.4 and not for Sections 4–6, provides a more detailed analysis of Theorem 1.2. For A and G as in (3), $C_n(A \rtimes G)$ has a subspace

$$C_n(A)_g := \text{span}\{ga_0 \otimes a_1 \otimes \cdots \otimes a_n : a_i \in A\}.$$

Notice that $(C_*(A)_g, b)$ is a subcomplex of $C_*(A \rtimes G, b)$. Its homology is

$$H_n(C_*(A)_g, b) = H_n(A, A_g),$$

where the A -bimodule structure on $M = A_g$ is given by $a \cdot m \cdot a' = g^{-1}(a)ma'$. According to [GeJo, Proposition 4.6]

$$HH_n(A \rtimes G) \cong H_n(\bigoplus_{g \in G} C_*(A)_g, b)_G \cong (\bigoplus_{g \in G} H_n(A, A_g))^G, \quad (9)$$

where the sub- and superscripts G mean coinvariants and invariants, respectively. The G -action is defined via the inclusion $C_*(A)_g \rightarrow C_*(A \rtimes G)$:

$$h \cdot (ga_0 \otimes a_1 \otimes \cdots \otimes a_n) = (hgh^{-1})h(a_0) \otimes h(a_1) \otimes \cdots \otimes h(a_n).$$

Moreover for $A = \mathcal{O}(X)$ and G as in Theorem 1.2, [Nis, Corollary 2.12] says that the inclusion $X^g \rightarrow X$ induces an isomorphism

$$H_n(\mathcal{O}(X), \mathcal{O}(X)_g) \cong HH_n(\mathcal{O}(X^g)) \cong \Omega^n(X^g). \quad (10)$$

The antisymmetrization map ϵ_n from [Lod, Section 1.3] gives a commutative diagram

$$\begin{array}{ccc} \Omega^n(X^g) & \longleftarrow & \Omega^n(\mathcal{O}(X)^g) \\ \downarrow \epsilon_n & & \downarrow \epsilon_n \\ HH_n(\mathcal{O}(X^g)) & \longleftarrow & HH_n(\mathcal{O}(X)^g), \end{array} \quad (11)$$

where $\Omega^n(B)$ denotes the module of Kähler n -forms over a commutative algebra B . Since X^g is nonsingular, the famous Hochschild–Kostant–Rosenberg theorem assures that the left ϵ_n is a bijection.

Since X^g is closed in X , the restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}(X^g)$ is surjective, and since g has finite order it admits a linear splitting $s : \mathcal{O}(X^g) \rightarrow \mathcal{O}(X)^g$. This induces a splitting

$$\Omega^n(s) : \Omega^n(X^g) \rightarrow \Omega^n(\mathcal{O}(X)^g)$$

of the upper line of (11). Now $\epsilon_n \circ \Omega^n(s) \circ \epsilon_n^{-1}$ in (11) defines a splitting of

$$HH_n(\mathcal{O}(X)^g) \rightarrow HH_n(\mathcal{O}(X^g)), \quad (12)$$

which therefore is surjective. We can embed $C_*(\mathcal{O}(X)^g)$ in $C_*(\mathcal{O}(X))_g$, simply by writing a g on the left. Together with (9), (10) and (12) this shows that the inclusion

$$\bigoplus_{g \in G} gC_*(\mathcal{O}(X)^g) \rightarrow C_*(\mathcal{O}(X) \rtimes G) \quad (13)$$

induces a surjection on homology. Moreover, because $HH_*(\mathcal{O}(X) \rtimes G)$ consists of G -coinvariants, it suffices to restrict the direct sum to one element in every conjugacy class of G .

Let $\langle g \rangle \subset G$ be the cyclic group generated by $g \in G$, so that g and $\mathcal{O}(X)^g$ lie in the commutative algebra $\mathcal{O}^g := \mathbb{C}[\langle g \rangle] \otimes \mathcal{O}(X)^g$. There is a natural surjection

$$\begin{aligned} \pi_n : C_n(\mathcal{O}^g) &\rightarrow \Omega^n(\mathcal{O}^g), \\ \pi_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 da_1 \cdots da_n. \end{aligned}$$

According to [Lod, Lemma 1.3.14 and Proposition 2.3.4] we have

$$\pi_n \circ b = 0 \quad \text{and} \quad \pi_{n+1} \circ B = (n+1) d \circ \pi_n.$$

Therefore $\pi_n/n!$ induces a map from $(\mathcal{B}_*(\mathcal{O}^g), b, B)$ to the mixed complex

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ \Omega^2(\mathcal{O}^g) & \xleftarrow{d} & \Omega^1(\mathcal{O}^g) & \xleftarrow{d} & \Omega^0(\mathcal{O}^g) \\ \downarrow 0 & & \downarrow 0 & & \\ \Omega^1(\mathcal{O}^g) & \xleftarrow{d} & \Omega^0(\mathcal{O}^g) & & \\ \downarrow 0 & & & & \\ \Omega^0(\mathcal{O}^g) & & & & \end{array} \quad (14)$$

Since $\Omega^n(\mathcal{O}(X)^g) \rightarrow \Omega^n(\mathcal{O}(X^g)) = \Omega^n(X^g)$ is surjective, the comparison of Theorem 1.2 with the homology of (14) shows that this gives us a surjection

$$HC_n(\mathcal{O}^g) \rightarrow HC_n(\mathcal{O}(X^g)).$$

As $\mathbb{C}[\langle g \rangle]$ barely plays a role here, the map remains surjective when we restrict it to elements coming from $g\mathcal{B}_*(\mathcal{O}(X)^g) \subset \mathcal{B}_*(\mathcal{O}^g)$. From this, Theorem 1.2 and (13) we deduce the result that we were after:

Lemma 1.3. *Let X and G be as in Theorem 1.2. All elements of $HH_n(\mathcal{O}(X) \rtimes G)$ and of $HC_n(\mathcal{O}(X) \rtimes G)$ can be represented by cycles in $\bigoplus_{g \in G} gC_*(\mathcal{O}(X)^g)$. It also suffices to take the direct sum over representatives of the conjugacy classes in G .*

2 Graded Hecke algebras

For the construction of graded Hecke algebras we will use the following objects:

- a finite dimensional real inner product space \mathfrak{a} ,
- the linear dual \mathfrak{a}^* of \mathfrak{a} ,

- a (reduced) root system R in \mathfrak{a}^* ,
- the dual root system R^\vee in \mathfrak{a} ,
- a basis Π of R .

We call

$$\tilde{\mathcal{R}} = (\mathfrak{a}^*, R, \mathfrak{a}, R^\vee, \Pi) \quad (15)$$

a degenerate root datum. We neither assume that R is crystallographic, nor that Π spans \mathfrak{a}^* . In fact R is even allowed to be empty. Our degenerate root datum gives rise to

- the complexifications \mathfrak{t} and \mathfrak{t}^* of \mathfrak{a} and \mathfrak{a}^* ,
- the symmetric algebra $S(\mathfrak{t}^*)$ of \mathfrak{t}^* ,
- the Weyl group W of R ,
- the set $S = \{s_\alpha : \alpha \in \Pi\}$ of simple reflections in W ,
- the complex group algebra $\mathbb{C}[W]$.

Choose formal parameters \mathbf{k}_α for $\alpha \in \Pi$, with the property that $\mathbf{k}_\alpha = \mathbf{k}_\beta$ if α and β are conjugate under W . The graded Hecke algebra $\tilde{\mathbb{H}}(\tilde{\mathcal{R}})$ corresponding to $\tilde{\mathcal{R}}$ is defined as follows. As a complex vector space

$$\tilde{\mathbb{H}}(\tilde{\mathcal{R}}) = \mathbb{C}[W] \otimes S(\mathfrak{t}^*) \otimes \mathbb{C}[\{\mathbf{k}_\alpha : \alpha \in \Pi\}].$$

The multiplication in $\tilde{\mathbb{H}}(\tilde{\mathcal{R}})$ is determined by the following rules:

- $\mathbb{C}[W]$, $S(\mathfrak{t}^*)$ and $\mathbb{C}[\{\mathbf{k}_\alpha : \alpha \in \Pi\}]$ are canonically embedded as subalgebras,
- the \mathbf{k}_α are central in $\tilde{\mathbb{H}}(\tilde{\mathcal{R}})$,
- for $x \in \mathfrak{t}^*$ and $s_\alpha \in S$ we have the cross relation

$$x \cdot s_\alpha - s_\alpha \cdot s_\alpha(x) = \mathbf{k}_\alpha \langle x, \alpha^\vee \rangle. \quad (16)$$

We define a grading on $\tilde{\mathbb{H}}(\tilde{\mathcal{R}})$ by requiring that \mathfrak{t}^* and the \mathbf{k}_α are in degree one, while W has degree zero.

In fact we will only study specializations of this algebra. Pick complex numbers $k_\alpha \in \mathbb{C}$ for $\alpha \in \Pi$, such that $k_\alpha = k_\beta$ if α and β are conjugate under W . Let \mathbb{C}_k be the onedimensional $\mathbb{C}[\{\mathbf{k}_\alpha : \alpha \in \Pi\}]$ -module on which \mathbf{k}_α acts as multiplication by k_α . We define

$$\mathbb{H} = \mathbb{H}(\tilde{\mathcal{R}}, k) = \tilde{\mathbb{H}}(\tilde{\mathcal{R}}) \otimes_{\mathbb{C}[\{\mathbf{k}_\alpha : \alpha \in \Pi\}]} \mathbb{C}_k \quad (17)$$

With some abuse of terminology $\mathbb{H}(\tilde{\mathcal{R}}, k)$ is also called a graded Hecke algebra. Notice that as a vector space $\mathbb{H}(\tilde{\mathcal{R}}, k)$ equals $\mathbb{C}[W] \otimes S(\mathfrak{t}^*)$, and that the cross relation (16) now holds with \mathbf{k}_α replaced by k_α :

$$x \cdot s_\alpha - s_\alpha \cdot s_\alpha(x) = k_\alpha \langle x, \alpha^\vee \rangle. \quad (18)$$

More generally, for any $s_\alpha \in S$ and $p \in S(\mathfrak{t}^*)$ we have

$$p \cdot s_\alpha - s_\alpha \cdot s_\alpha(p) = k_\alpha \frac{p - s_\alpha(p)}{\alpha}. \quad (19)$$

An automorphism γ of the Dynkin diagram of the based root system (R, Π) is a bijection $\Pi \rightarrow \Pi$ such that

$$\langle \gamma(\alpha), \gamma(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \quad \forall \alpha, \beta \in \Pi. \quad (20)$$

Such a γ naturally induces automorphisms of R, R^\vee and W . Moreover we will assume that the action of γ on the linear span of the coroots is extended somehow to an orthogonal endomorphism of \mathfrak{t} . If γ and γ' act in the same way on R but differently on \mathfrak{t} , then we will sloppily regard them as different diagram automorphisms.

Let Γ be a finite group of diagram automorphisms of (R, Π) . Groups like

$$W' := \Gamma \ltimes W \quad (21)$$

typically arise from larger Weyl groups as the isotropy groups of points in some torus, or as normalizers of some parabolic subgroup [How]. Assume that $k_{\gamma(\alpha)} = k_\alpha \forall \alpha \in \Pi, \gamma \in \Gamma$. Then Γ acts on \mathbb{H} by the algebra homomorphisms

$$\begin{aligned} \psi_\gamma : \mathbb{H} &\rightarrow \mathbb{H}, \\ \psi_\gamma(xs_\alpha) &= \gamma(x)s_{\gamma(\alpha)} \quad x \in \mathfrak{t}^*, \alpha \in \Pi. \end{aligned} \quad (22)$$

Thus we can form the crossed product

$$\mathbb{H}' := \Gamma \ltimes \mathbb{H} = \Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k), \quad (23)$$

which we call an extended graded Hecke algebra.

We define a \mathbb{Z} -grading on \mathbb{H}' by $\deg(w) = 0 \forall w \in W'$ and $\deg(x) = 1 \forall x \in \mathfrak{t}^*$. However, the algebra \mathbb{H}' is in general not graded, only filtered. That is, the product $h_1 h_2$ of two homogeneous elements $h_1, h_2 \in \mathbb{H}'$ need not be homogeneous, but all its homogeneous components have degree at most $\deg(h_1) + \deg(h_2)$. More precisely, from (19) we see that the part of $h_1 h_2$ that depends on the parameters k_α has degree strictly lower than $\deg(h_1) + \deg(h_2)$.

Let us mention some special cases in which $\mathbb{H}' = \Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)$ is graded:

- if $R = \emptyset$ then $\mathbb{H}' = \Gamma \ltimes \tilde{\mathbb{H}}(\tilde{\mathcal{R}}) = \Gamma \ltimes S(\mathfrak{t}^*)$,
- if $k_\alpha = 0 \forall \alpha \in \Pi$ then $\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)$ is the crossed product $W' \ltimes S(\mathfrak{t}^*)$, with the cross relations

$$w \cdot x = w(x) \cdot w \quad w \in W', x \in \mathfrak{t}^*. \quad (24)$$

Multiplication with any $z \in \mathbb{C}^\times$ defines a bijection $m_z : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$, which clearly extends to an algebra automorphism of $S(\mathfrak{t}^*)$. From the cross relation (18) we see that it extends even further, to an algebra isomorphism

$$m_z : \Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, zk) \rightarrow \Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k) \quad (25)$$

which is the identity on $\mathbb{C}[W']$. Notice that the homomorphism (25) remains well-defined for $z = 0$, only then it ceases to be bijective.

In particular, if all $\alpha \in R$ are conjugate under W' , then there are essentially only two graded Hecke algebras attached to $(\tilde{\mathcal{R}}, \Gamma)$: one with $k = 0$ and one with $k \neq 0$.

3 The homology of graded Hecke algebras

In this section we will prove that the periodic cyclic homology of a graded Hecke algebra is the same as that of its underlying finite Weyl group. In fact we immediately generalize this to the extended graded Hecke algebras from (23), since this does not make the proof more difficult. Throughout we will use the notations from Section 2.

Lemma 3.1. *Consider the extended graded Hecke algebra $\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, 0) = W' \ltimes S(\mathfrak{t}^*)$ with parameter $k = 0$.*

- a) $HH_n(W' \ltimes S(\mathfrak{t}^*)) = 0$ for all $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$.
- b) $HC_n(W' \ltimes S(\mathfrak{t}^*)) = HP_n(W' \ltimes S(\mathfrak{t}^*)) = 0$ for all odd $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$.
- c) *The inclusion $\mathbb{C}[W'] \rightarrow W' \ltimes S(\mathfrak{t}^*)$ induces an isomorphism on periodic cyclic homology.*

Proof. a) follows immediately from Theorem 1.2.

b) Every component of $\tilde{\mathfrak{t}}$ is a vector space, and in particular contractible as a topological space. Therefore $H_{DR}^n(\tilde{\mathfrak{t}}) = 0$ for all $n > 0$. Now apply Theorem 1.2.

c) By the above we only have to show this in degree 0. Again by Theorem 1.2

$$\begin{aligned} HP_0(W' \ltimes S(\mathfrak{t}^*)) &\cong H_{DR}^0(\tilde{\mathfrak{t}})^{W'} \cong \left(\bigoplus_{w \in W'} H_{DR}^0(\mathfrak{t}^w) \right)^{W'} \\ &\cong (\mathbb{C}[W'])^{W'} \cong HP_0(\mathbb{C}[W'])^{W'}. \end{aligned}$$

According to [Nis, Theorem 2.11] all the above isomorphisms can be made natural. In particular $\mathbb{C}[W'] \rightarrow W' \ltimes S(\mathfrak{t}^*)$ induces an isomorphism on HP_* . \square

To transfer these results to $\mathbb{H}' = \Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)$ with $k \neq 0$, we will use spectral sequences. For $d \in \mathbb{Z}_{\geq 0}$ let $\mathbb{H}'_{\leq d}$ denote the vector space of elements of degree $\leq d$ in \mathbb{H}' . We generalize this filtration to the tensor power $(\mathbb{H}')^{\otimes n}$ by

$$(\mathbb{H}')_{\leq d}^{\otimes n} := \sum_{d_1 + \dots + d_n \leq d} \mathbb{H}'_{\leq d_1} \otimes \dots \otimes \mathbb{H}'_{\leq d_n}. \quad (26)$$

Equivalently one can take the sum over all n -tuples (d_1, \dots, d_n) of nonnegative integers with $d_1 + \dots + d_n = d$.

Lemma 3.2. $HH_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) = 0$ for $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$.

Proof. The Hochschild complex $(C_*(\mathbb{H}'), b)$ from (6) is filtered by (26). That is, we put

$$F_p C_n(\mathbb{H}') = (\mathbb{H}')_{\leq p}^{\otimes (n+1)}.$$

This gives rise to a spectral sequence converging to $HH_*(\mathbb{H}')$, see for example [CaEi, Chapter XV]. Its first term is

$$E_{p,q}^1 = H_{p+q}(F_p C_*(\mathbb{H}') / F_{p-1} C_*(\mathbb{H}')).$$

The boundary map on $F_p C_*(\mathbb{H}')/F_{p-1} C_*(\mathbb{H}')$ is given by (6), but we may ignore all terms that are not in top degree. From equations (18) and (22) we see that the resulting map is actually independent of k . So we can determine $E_{p,q}^1$ for all parameters, just by looking at the case $k = 0$.

But for $k = 0$ the algebra $\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, 0) = W' \ltimes S(\mathfrak{t}^*)$ is really graded, so filtering the Hochschild complex does not add anything, and the spectral sequence stabilizes already at $E_{*,*}^1$. From Lemma 3.1.a we deduce that $E_{p,q}^1 = 0$ if $p + q > \dim_{\mathbb{C}}(\mathfrak{t}^*)$.

For general parameters k_α we cannot say immediately whether our spectral sequence stabilizes at the first term, but in any case $E_{p,q}^r$ is a subquotient of $E_{p,q}^1$. Hence $E_{p,q}^\infty = 0$ whenever $p + q > \dim_{\mathbb{C}}(\mathfrak{t}^*)$, and the lemma follows. \square

As a corollary of Lemma 3.2 we find that for $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$:

$$HC_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) \cong HP_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)). \quad (27)$$

Indeed, this can be seen from Connes' periodicity exact sequence, which relates Hochschild and cyclic homology. On the other hand, we can also try to compute the cyclic homology with a spectral sequence, and this leads to the following result.

Theorem 3.3. *For $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$ we have*

$$HP_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) \cong HC_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) \cong HC_n(\mathbb{C}[\Gamma \ltimes W]).$$

The inclusion $\mathbb{C}[\Gamma \ltimes W] \rightarrow \Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)$ induces an isomorphism on periodic cyclic homology.

Proof. Recall from (7) that $HC_n(\mathbb{H}')$ is computed by the mixed complex $(\mathcal{B}_*(\mathbb{H}'), b, B)$. Its space of n -chains is

$$\mathcal{B}_n(\mathbb{H}') := \mathbb{H}'^{\otimes(n+1)} \oplus \mathbb{H}'^{\otimes n} \oplus \dots \oplus \mathbb{H}'.$$

We introduce a filtration on this differential complex by

$$F_p \mathcal{B}_n(\mathbb{H}') := (\mathbb{H}')_{\leq p}^{\otimes(n+1)} \oplus (\mathbb{H}')_{\leq p}^{\otimes n} \oplus \dots \oplus \mathbb{H}'_{\leq p}.$$

As in the previous proof, this yields a spectral sequence $E_{p,q}^r$ converging to $HC_{p+q}(\mathbb{H}')$. In particular

$$E_{p,q}^1 = H_{p+q}(F_p \mathcal{B}_*(\mathbb{H}')/F_{p-1} \mathcal{B}_*(\mathbb{H}')), \quad (28)$$

By (18) and (22) the boundary maps in $F_p \mathcal{B}_*(\mathbb{H}')/F_{p-1} \mathcal{B}_*(\mathbb{H}')$ are independent of the parameters k_α . Hence the vector spaces $E_{p,q}^1$ do not depend on k , and we can determine them from the special case $k = 0$.

Because the algebra $\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, 0) = W' \ltimes S(\mathfrak{t}^*)$ is graded, the spectral sequence for this algebra already stabilizes at $E_{*,*}^1$, and $E_{p,q}^1$ is just the degree p part of $HC_{p+q}(W' \ltimes S(\mathfrak{t}^*))$. Unlike Hochschild homology, $HC_n(W' \ltimes S(\mathfrak{t}^*))$ does not vanish for (sufficiently) large n , but from Lemma 3.1.b we do know that $HC_n(W' \ltimes S(\mathfrak{t}^*)) = 0$ when $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$ is odd. Moreover, for even $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$ Theorem 1.2 says that

$$HC_n(W' \ltimes S(\mathfrak{t}^*)) = HP_n(W' \ltimes S(\mathfrak{t}^*)).$$

It follows from Lemma 3.1.c that, for all $n > \dim_{\mathbb{C}}(\mathfrak{t}^*)$, $HC_n(W' \ltimes S(\mathfrak{t}^*))$ has no parts in degrees $p > 0$, and that its degree $p = 0$ part is $HH_0(\mathbb{C}[W'])$.

Now we return to general k and consider various p and q . By definition $E_{p,q}^r = 0$ if $p < 0$ or $p+q < 0$. We can not yet say much about $E_{p,q}^1$ when $0 \leq p+q \leq \dim_{\mathbb{C}}(\mathfrak{t}^*)$, but we can do without.

Finally, we pick $p, q \in \mathbb{Z}$ such that $p+q > \dim_{\mathbb{C}}(\mathfrak{t}^*)$. By the above $E_{p,q}^1 = 0$ unless $p = 0$ and q is even, in which case $E_{p,q}^1 = HH_0(\mathbb{C}[W'])$. For every $r \in \mathbb{Z}_{\geq 0}$ there is a boundary map $\partial_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$, and $E_{*,*}^{r+1}$ is the homology of $(E_{*,*}^r, \partial_{*,*}^r)$. We claim that $\partial_{p,q}^r = 0$ whenever $r \geq 1$ and $p+q > \dim_{\mathbb{C}}(\mathfrak{t}^*)$. Indeed, for $p > 0$ the domain is zero, while for $p = 0$ the range is $E_{-r,q+r-1}^r$, which is zero because $-r < 0$.

We conclude that in the range $p+q > \dim_{\mathbb{C}}(\mathfrak{t}^*)$ our spectral sequence stabilizes at $r = 1$, for all k . Moreover, we already showed that the vector spaces $E_{p,q}^1$ do not depend on k . Taking the limit $r \rightarrow \infty$ and using the convergence, we find that $HC_{p+q}(\mathbb{H}')$ does not depend on k . In view of (27) the same goes for $HP_{p+q}(\mathbb{H}')$. But the functor HP_* is 2-periodic, so actually $HP_n(\mathbb{H}')$ is independent of k for all $n \in \mathbb{Z}$. Together with Lemma 3.1.c this shows that

$$HP_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) \cong HP_n(\mathbb{C}[\Gamma \ltimes W]) \quad \forall n \in \mathbb{Z}.$$

Furthermore we noticed that for large n the representatives of

$$HC_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) \cong HP_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k))$$

all lie in the image of

$$HP_n(\mathbb{C}[\Gamma \ltimes W]) \rightarrow HP_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)). \quad (29)$$

Therefore (29) is a linear bijection. \square

Our next result is a clear improvement on Lemma 3.2 and Theorem 3.3. Although the proofs of these three results could have been combined in one, we decided against this. The previous results are relatively general and the proofs probably can be applied to similar algebras as well. The upcoming result however uses more subtle properties of root systems and crossed products.

Theorem 3.4. *For all $n \in \mathbb{Z}_{\geq 0}$ there are isomorphisms*

$$\begin{aligned} HH_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) &\cong HH_n(W' \ltimes S(\mathfrak{t}^*)), \\ HC_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) &\cong HC_n(W' \ltimes S(\mathfrak{t}^*)), \\ HP_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) &\cong HP_n(W' \ltimes S(\mathfrak{t}^*)). \end{aligned}$$

Proof. In the proofs of Lemma 3.2 and Theorem 3.3 we constructed spectral sequences $E_{p,q}^r$ converging to $HH_{p+q}(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k))$ and $HC_{p+q}(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k))$, respectively. We showed that the spaces $E_{p,q}^1$ do not depend on k , but it is conceivable that boundary maps $\partial_{p,q}^r$ do. With a clever choice of representatives for the homology classes we will show that $\partial_{p,q}^r = 0$ for all $r \geq 1$. We will only write this down for

HC_* , the Hochschild homology can be handled in the same way. Recall from page 12 that

$$\bigoplus_{p+q=n} E_{p,q}^1 \cong HC_n(W' \ltimes S(\mathfrak{t}^*)). \quad (30)$$

By Lemma 1.3 every element of $HC_n(W' \ltimes S(\mathfrak{t}^*))$ can be represented by a cycle in

$$\bigoplus_{g \in W'} gC_*(\mathcal{O}(\mathfrak{t})^g) = \bigoplus_{g \in W'} g \bigoplus_{m \geq 1} (\mathcal{O}(\mathfrak{t})^g)^{\otimes m}.$$

Moreover we need only one g from every conjugacy class in W' , and the map from cycles to homology classes factors through $\mathcal{O}(\mathfrak{t})^g \rightarrow \mathcal{O}(\mathfrak{t}^g)$.

Recall that the closed Weyl chamber

$$\mathfrak{a}^+ := \{\lambda \in \mathfrak{a} : \langle \alpha, \lambda \rangle \geq 0 \ \forall \alpha \in \Pi\}$$

is a fundamental domain for the action of W on \mathfrak{a} . Take any $g \in W'$ and write

$$\begin{aligned} R_g &= \{\alpha \in R : \alpha \perp \mathfrak{a}^g\}, \\ \mathfrak{a}^{R_g} &= \{\lambda \in \mathfrak{a} : \langle \alpha, \lambda \rangle = 0 \ \forall \alpha \in R_g\}. \end{aligned}$$

Clearly $\mathfrak{a}^g \subset \mathfrak{a}^{R_g}$. By [Hum, Theorem 1.12] there exists $u \in W$ such that

$$u(\mathfrak{a}^{R_g}) = \mathfrak{a}^P := \{\lambda \in \mathfrak{a} : \langle \alpha, \lambda \rangle = 0 \ \forall \alpha \in P\}$$

for some set $P \subset \Pi$ of simple roots. Replacing g by ugu^{-1} , we may assume that $\mathfrak{a}^g \subset \mathfrak{a}^P$. Then

$$\mathfrak{a}^g = (\mathfrak{a}^g \cap \mathfrak{a}^+) + (\mathfrak{a}^g \cap -\mathfrak{a}^+).$$

Since Γ permutes Π , it stabilizes \mathfrak{a}^+ . So if we write $g = \gamma w$ with $\gamma \in \Gamma$ and $w \in W$, then

$$w(\mathfrak{a}^g \cap \mathfrak{a}^+) = \gamma^{-1}(\mathfrak{a}^g \cap \mathfrak{a}^+) \subset \mathfrak{a}^+.$$

Hence w must fix $\mathfrak{a}^g \cap \mathfrak{a}^+$, and we deduce that both w and γ fix \mathfrak{a}^g pointwise. Moreover by [Hum, Theorem 1.12] w lies in the parabolic subgroup $W_P \subset W$. Let $\mathfrak{t}_P \subset \mathfrak{t}$ be the complex span of $\{\alpha^\vee : \alpha \in P\}$. This is a complement to $\mathfrak{t}^P = \mathfrak{a}^P + i\mathfrak{a}^P$ in \mathfrak{t} and it is stable under γ and W_P . Clearly the natural map $\mathcal{O}(\mathfrak{t}/\mathfrak{t}_P) \rightarrow \mathcal{O}(\mathfrak{t}^g)$ is surjective. Hence every element of $HC_n(W' \ltimes S(\mathfrak{t}^*))$ can be represented by a cycle

$$z \in \bigoplus_g gC_*(\mathcal{O}(\mathfrak{t}/\mathfrak{t}_P)),$$

where we sum only over $g = \gamma w$ as above. Now consider z as an element of the mixed complex $\mathcal{B}_*(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k))$. Since γ stabilizes \mathfrak{t}_P and all the simple reflections s_α with $\alpha \in P$ act trivially on $\mathfrak{t}/\mathfrak{t}_P$, the multiplication of $g = \gamma w \in \Gamma W_P$ with elements of $\mathcal{O}(\mathfrak{t}/\mathfrak{t}_P)$ does not depend on k . Consequently z is also a cycle in $\mathcal{B}_*(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k))$ and $\partial_{*,*}^r(z) = 0$ for all $r \geq 0$.

By (30) every element of $E_{*,*}^1$ can be obtained in this way, so $\partial_{*,*}^r = 0$ for all $r \geq 1$ and the spectral sequence $E_{*,*}^r$ stabilizes at $r = 1$. This proves the theorem for HH_* and HC_* , the statement for HP_* was already contained in Theorem 3.3. \square

We remark that

$$HH_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k)) \cong HH_n(W' \ltimes S(\mathfrak{t}^*))$$

is not an isomorphism of $S(\mathfrak{t}^*)^{W'}$ -modules. For $k \neq 0$ this structure is distorted in the process of choosing suitable representatives of homology classes. However, it is still a little more than just a linear bijection between vector spaces. The direct summand $\Omega^n(\mathfrak{t}^g)^{Z_{W'}(g)}$ of $HH_n(W' \ltimes S(\mathfrak{t}^*))$ is a module over $\mathcal{O}(\mathfrak{t}^g)^{Z_{W'}(g)} \subset S(\mathfrak{t}^*)^{W'}$, and this module structure persists to the corresponding summand of $HH_n(\Gamma \ltimes \mathbb{H}(\tilde{\mathcal{R}}, k))$.

4 Periodic cyclic homology as sheaf cohomology

To draw representation theoretic consequences from knowledge of the periodic cyclic homology of an algebra, we will relate it to its primitive ideal spectrum. This is possible for algebras of finite type, as studied in [KNS, BaNi].

Let X be a complex, affine, algebraic variety and $\mathcal{O}(X)$ its coordinate ring. A finite type $\mathcal{O}(X)$ -algebra is an algebra A together with a morphism from $\mathcal{O}(X)$ to the center of the multiplier algebra of A , which makes A into an $\mathcal{O}(X)$ -module of finite rank. An algebra is said to be of finite type if it is a finite type $\mathcal{O}(X)$ -algebra for some X . Notice that a unital finite type algebra is always of finite type over its center.

Recall that a primitive ideal of an algebra A is the annihilator of an irreducible A -module. We endow the collection $\text{Prim}(A)$ of primitive ideals with the Jacobson topology, whose closed sets are of the form

$$V(S) := \{I \in \text{Prim}(A) : S \subset I\},$$

for any subset $S \subset A$. The resulting topological space is called the primitive ideal spectrum of A . Notice that $\text{Prim}(\mathcal{O}(X))$ is X with the Zariski topology.

Lemma 4.1. *The primitive ideal spectrum of a finite type algebra is a T_1 -space.*

Proof. Recall that a topological space is T_1 if and only if every one point subset is closed. Let A be (not necessarily unital) finite type $\mathcal{O}(X)$ -algebra, let V be an irreducible A -module and $J \in \text{Prim}(A)$ the annihilator of V . We want to show that $V(J) = \{J\}$.

The A -module structure on V extends naturally to the multiplier algebra $\mathcal{M}(A)$, because $V = AV$. Let $\phi_A : \mathcal{O}(X) \rightarrow \mathcal{M}(A)$ be the morphism that defines the $\mathcal{O}(X)$ -algebra structure on A . It enables us to construct the unital finite $\mathcal{O}(X)$ -algebra $A^+ := A \oplus \mathcal{O}(X)$ with multiplication

$$(a, f)(a', f') = (aa' + \phi_A(f)a' + a\phi_A(f'), ff').$$

The A -module structure on V extends to A^+ via

$$(a, f)v := (a + \phi_A(f))v.$$

Since A^+ has (at most) countable dimension over \mathbb{C} , the dimension of V is countable or finite. Therefore Schur's lemma applies, and it tells us that $Z(A^+)$ acts on V via a character $\chi : Z(A^+) \rightarrow \mathbb{C}$. Since A^+ is a unital finite type algebra, it has finite rank over $Z(A^+)$, and consequently the image of $A^+ \rightarrow \text{End}_{\mathbb{C}}(V)$ has finite dimension. But V is irreducible, so $\dim(V)$ has to be finite. If J^+ is the annihilator of V in A^+ , then Wedderburn's theorem says that $A^+/J^+ \rightarrow \text{End}_{\mathbb{C}}(V)$ is an isomorphism.

Suppose that there is a primitive ideal I of A such that $I \supsetneq J$. Then $(I + J^+)/J^+$ is a nonzero ideal of the simple algebra $A^+/J^+ \cong \text{End}_{\mathbb{C}}(V)$, so $(I + J^+)/J^+ = A^+/J^+$. But then

$$A = (I + J^+) \cap A = I + (J^+ \cap A) = I + J = I,$$

which contradicts the primitivity of I . We conclude that the subset $V(J) \subset \text{Prim}(A)$, which by definition is closed, equals $\{J\}$. \square

For any unital finite algebra A with center $\mathcal{O}(X)$ the map

$$\begin{aligned} \theta : \text{Prim}(A) &\rightarrow \text{Prim}(\mathcal{O}(X)) \cong X, \\ \theta(I) &= I \cap \mathcal{O}(X) \end{aligned} \tag{31}$$

is a finite-to-one continuous surjection [KNS, Lemma 1]. In other words, $\text{Prim}(A)$ may be complicated, but cannot be too different from an algebraic variety. The periodic cyclic homology of commutative finite type algebras was determined in [KNS, Theorem 9]:

Theorem 4.2. *Let $I \subset \mathcal{O}(X)$ be an ideal and $Y \subset X$ its zero locus. There is a natural isomorphism*

$$HP_n(I) \cong \bigoplus_{m \in \mathbb{Z}} \check{H}^{n+2m}(X^{an}, Y^{an}; \mathbb{C}).$$

Here the superscript “an” means that we endow this variety with the analytic topology, and \check{H}^* is (relative) Čech-cohomology.

It turns out that the periodic cyclic homology of a general finite type algebra depends only on its primitive ideal spectrum. Following [BaNi] we call a morphism $\phi : A \rightarrow B$ of finite type $\mathcal{O}(X)$ -algebras spectrum preserving if:

- for each $J \in \text{Prim}(B)$ there is exactly one $I \in \text{Prim}(A)$ containing $\phi^{-1}(J)$,
- the map $\text{Prim}(B) \rightarrow \text{Prim}(A) : J \mapsto I$ is a bijection.

Theorem 4.3. [BaNi, Theorem 8]

Let $\phi : A \rightarrow B$ be a spectrum preserving morphism of finite type $\mathcal{O}(X)$ -algebras. Then $HP_(\phi) : HP_*(A) \rightarrow HP_*(B)$ is an isomorphism.*

In view of these two Theorems, $HP_*(A)$ can be regarded as a kind of cohomology of $\text{Prim}(A)$. But it requires some care to describe exactly how $HP_*(A)$ can be determined from the primitive ideal spectrum of A . Let $\text{Prim}(A)^{an}$ be the set $\text{Prim}(A)$ with the coarsest topology that makes

$$\text{Prim}(A)^{an} \rightarrow \text{Prim}(A) \times X^{an} : I \mapsto (I, \theta(I))$$

continuous. In this space every point has an open neighborhood homeomorphic to $\mathbb{R}^n \times \mathbb{R}_{\geq 0}^m$ for some $n, m \in \mathbb{Z}_{\geq 0}$, so roughly speaking it is a non-Hausdorff version of an orbifold.

If we would use a classical cohomology theory on $\text{Prim}(A)^{an}$, the homotopy axiom would imply that we cannot see the entire space, only some Hausdorff quotient. The solution is to consider not $\text{Prim}(A)^{an}$ but X^{an} , and to encode the additional structure of $\text{Prim}(A)$ in a sheaf S_A over X^{an} . This dictates that the stalk $S_A(x)$ must be the complex vector space $\mathbb{C}\theta^{-1}(x)$, whose basis can be identified with the inequivalent irreducible A -modules with $\mathcal{O}(X)$ -character $x \in X$.

Sections of this sheaf will be modelled by idempotents in suitable completions of the algebra A . More precisely, given an open $U \subset X^{an}$, pick a connected component C of $\theta^{-1}(U) \subset \text{Prim}(A)^{an}$, and let C^H be its maximal Hausdorff quotient. We note that $\theta : C \rightarrow U$ factors as

$$C \xrightarrow{q_C} C^H \xrightarrow{\theta_C} U.$$

Let $s_C : C^H \rightarrow C$ be a section of q_C and put

$$\begin{aligned} s : U &\rightarrow \bigsqcup_{x \in U} S_A(x), \\ s(x) &= \sum_{c \in C^H : \theta_C(c) = x} s_C(c). \end{aligned} \tag{32}$$

Definition 4.4. *Let A be a finite type $\mathcal{O}(X)$ -algebra. Then S_A is the sheaf over X^{an} with stalks $S_A(x) = \mathbb{C}\theta^{-1}(x)$, whose continuous sections can be written locally as finite sums $\sum_i z_i s_i$, with $z_i \in \mathbb{C}$ and s_i of the form (32).*

This construction is functorial in A . Indeed, suppose that $\phi : A \rightarrow B$ is a morphism of finite type $\mathcal{O}(X)$ -algebras. Then $V \mapsto B \otimes_A V$ is a functor from left A -modules to left B -modules, and it induces a map $S_A \rightarrow S_B$ of sheaves over X .

Examples.

Suppose that $A \subset \mathcal{O}(X) \otimes \text{End}_{\mathbb{C}}(V)$ is unital and $e \in A$ is idempotent. Then we can get a global section $s_e \in S_A(X)$ as follows. For any $x \in X$ let

$$0 \subset V_x^1 \subset \cdots \subset V_x^d = V_x$$

be a composition series of the A -module V_x , the vector space V on which $\mathcal{O}(X)$ acts by x . For a primitive ideal $I \in \theta^{-1}(x)$ let

$$n(e, I) = \sum_m \text{rank}(e, V_x^m / V_x^{m-1}) \in \mathbb{Z}_{\geq 0},$$

where the sum runs over all m for which $I \subset A$ is the annihilator of the irreducible module V_x^m / V_x^{m-1} . Then

$$s_e(x) := \sum_{I \in \theta^{-1}(x)} n(e, I) I \tag{33}$$

defines a continuous section of S_A .

For a more concrete example, fix a point $x_0 \in X$ and consider the algebra

$$B := \{b \in \mathcal{O}(X) \otimes M_2(\mathbb{C}) : b(x_0) \text{ is a diagonal matrix}\}.$$

Then $\text{Prim}(B)$ is X , but with two points x'_0, x''_0 instead of x_0 . The topology of $\text{Prim}(B)^{an}$ is such that every sequence $(x_n)_{n=1}^\infty$ in $X \setminus \{x_0\}$, which converges to x_0 in X^{an} , converges to both x'_0 and x''_0 in $\text{Prim}(B)^{an}$. The stalk $S_B(x)$ is canonically isomorphic to \mathbb{C} for $x \in X \setminus \{x_0\}$, while $S_B(x_0) \cong \mathbb{C}^2$. In particular the sheaf S_B on X^{an} is not a vector bundle. A global section s of S_A is continuous if and only if the function

$$X^{an} \rightarrow \mathbb{C} : x \mapsto \begin{cases} s(x) & \text{if } x \in X \setminus \{x_0\} \\ z_1 + z_2 & \text{if } x = x_0 \text{ and } s(x_0) = (z_1, z_2) \end{cases}$$

is locally constant. Thus the continuous sections with $s(x_0) = (z_1, 0)$ form a copy of the constant sheaf (with stalks \mathbb{C}) on X^{an} . Furthermore there is a continuous section s_0 with

$$s(x) = \begin{cases} 0 & \text{if } x \in X \setminus \{x_0\} \\ (-1, 1) & \text{if } x = x_0. \end{cases}$$

These specific sections provide an isomorphism between S_B and the direct sum of the constant sheaf on X^{an} and the skyscraper sheaf concentrated at x_0 . Hence the sheaf cohomology $\check{H}^*(X^{an}; S_B)$ is isomorphic to

$$\check{H}^*(X^{an}; \mathbb{C}) \oplus \check{H}^*(\{x_0\}; \mathbb{C}).$$

Let us compare this to $HP_*(B)$. We abbreviate $\check{H}^{ev} = \bigoplus_{m \in \mathbb{Z}} \check{H}^{2m}$ and $\check{H}^{odd} = \bigoplus_{m \in \mathbb{Z}} \check{H}^{2m+1}$. Applying Theorem 4.3 and the excision property of HP_* to the extension

$$0 \rightarrow \{f \in \mathcal{O}(X) : f(x_0) = 0\} \otimes M_2(\mathbb{C}) \rightarrow B \rightarrow \mathcal{O}(\{x_0\}) \otimes \mathbb{C}^2 = \mathbb{C}^2 \rightarrow 0$$

we find an exact hexagon

$$\begin{array}{ccccc} \check{H}^{ev}(X, \{x_0\}; \mathbb{C}) & \rightarrow & HP_0(B) & \rightarrow & \check{H}^{ev}(\{x_0\}; \mathbb{C}^2) \cong \mathbb{C}^2 \\ \uparrow & & & & \downarrow \\ \check{H}^{odd}(\{x_0\}; \mathbb{C}^2) = 0 & \leftarrow & HP_1(B) & \leftarrow & \check{H}^{odd}(X, \{x_0\}; \mathbb{C}) \end{array} \quad (34)$$

By comparing (34) with the hexagon associated to the algebra extension

$$0 \rightarrow \{f \in \mathcal{O}(X) : f(x_0) = 0\} \rightarrow \mathcal{O}(X) \rightarrow \mathcal{O}(\{x_0\}) = \mathbb{C} \rightarrow 0$$

one sees that the right vertical map in (34) is zero. We deduce that

$$HP_0(B) \cong \check{H}^{ev}(X^{an}; \mathbb{C}) \oplus \mathbb{C} \quad \text{and} \quad HP_1(B) \cong \check{H}^{odd}(X^{an}; \mathbb{C}).$$

The resulting isomorphism between $HP_*(B)$ and the sheaf cohomology of S_B is not quite canonical, since it depends on the decomposition of S_B into a constant sheaf and a skyscraper sheaf.

In general, let $\check{H}^p(X^{an}; S_A)$ denote the p -th Čech-cohomology group of the sheaf S_A . Since the space X^{an} is so nice, there is no need to distinguish between the Čech-cohomology and the sheaf cohomology of S_A .

Theorem 4.5. *Let A be a finite type $\mathcal{O}(X)$ -algebra. There exists a spectral sequence converging to both $HP_n(A)$ and $\bigoplus_{m \in \mathbb{Z}} \tilde{H}^{n+2m}(X^{an}; S_A)$. In particular these finite dimensional vector spaces are (unnaturally) isomorphic.*

Proof. Using the “abelian filtrations” of A studied in [KNS], we can set up and compare two spectral sequences, one converging to $HP_*(A)$ and the other to $H^*(X^{an}; S_A)$. They differ in level zero, but agree on all higher levels. The details can be found in the author’s PhD thesis [Sol1, Section 2.2]. \square

With Theorem 4.5 one can determine $HP_*(A)$ from $\text{Prim}(A)$ by geometric means. In particular it becomes unnecessary to consider ideals as in the abelian filtration of A , which can be hard to determine explicitly.

Examples.

First the commutative case, where I is an ideal of $\mathcal{O}(X)$ and $Y \subset X$ is the zero locus of I . Then S_I is the direct image, with respect to the inclusion map $X \setminus Y \rightarrow X$, of the constant sheaf with fiber \mathbb{C} on $X \setminus Y$. Thus it appears that we recover Theorem 4.2 from Theorem 4.5, but in fact the former is an essential ingredient in the proof of the latter.

Next suppose that a finite group G acts on the affine variety X , and consider the finite type $\mathcal{O}(X/G)$ -algebra $B := \mathcal{O}(X) \rtimes G$. Using Theorem 1.1.e one can construct a continuous bijection from the extended quotient \tilde{X}/G to $\text{Prim}(\mathcal{O}(X) \rtimes G)$. We note that in general this map is neither canonical nor an homeomorphism.

Consider the constant sheaf with fiber \mathbb{C} on \tilde{X}/G . We claim that the direct image \mathcal{F} of this sheaf under the natural map $\tilde{X}/G \rightarrow X/G$ is isomorphic to the sheaf S_B . By Theorem 1.1.e both have isomorphic stalks over any point of X/G . To construct a sheaf map $\mathcal{F} \rightarrow S_B$, consider any section f of \mathcal{F} . For $Gx \in X/G$ in the domain of f there is a unique virtual G_x -representation $\pi_x \in R(G_x) \otimes_{\mathbb{Z}} \mathbb{C} \cong S_B(Gx)$ with trace $\text{tr}(\pi_x, g) = f(g, x)$. We define a section s of S_B by $s(x) = \pi_x$. Now $f \mapsto s$ is a sheaf map which induces bijections on the stalks, so it is an isomorphism $\mathcal{F} \rightarrow S_B$.

For later use we will now calculate the cohomology of certain sheaves of the type S_A . Let X be a countable, locally finite, finite dimensional CW-complex space endowed with an action of a finite group G . Let V be a finite dimensional complex vector space and put $A = C(X) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(V)$. Suppose furthermore that we have elements $u_g \in A^\times$ such that

$$(g \cdot a)(x) := u_g(x) a(g^{-1}x) u_g^{-1}(x) \quad g \in G, a \in A, x \in X$$

defines an action of G on A by algebra automorphisms. In particular, for every $x \in X$ we get a projective G_x -representation V_x , by $g \mapsto u_g(x) \in PGL(V)$.

The algebra A^G of G -invariant elements in A can be considered as a topological analogue of a finite type algebra. Its topological K -theory $K_*(A^G)$ behaves much like the periodic cyclic homology of finite type algebras. Indeed, we can define a sheaf S_{A^G} over X/G in the same way as in Definition 4.4. According to [Sol1,

Theorem 2.24] there is an isomorphism

$$K_n(A^G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{m \in \mathbb{Z}} \check{H}^{n+2m}(X/G; S_{A^G}), \quad (35)$$

if at least one side has finite dimension.

Lemma 4.6. *Suppose that X is G -equivariantly contractible to a point $x_0 \in X$. Then*

$$\check{H}^n(X/G; S_{A^G}) = 0 \quad \forall n > 0$$

and the evaluation at x_0 induces a linear bijection

$$\check{H}^0(X/G; S_{A^G}) \rightarrow S_{A^G}(x_0) \cong HH_0(\text{End}_G(V_{x_0})).$$

Proof. By [Sol1, Lemma 2.26] the evaluation map $A^G \rightarrow \text{End}_G(V_{x_0})$ is a homotopy equivalence. In particular it induces an isomorphism on topological K -theory, so by (35)

$$\bigoplus_{m \in \mathbb{Z}} \check{H}^{n+2m}(X/G; S_{A^G}) \cong K_n(A^G) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_n(\text{End}_{\mathbb{C}}(V_{x_0})) \otimes_{\mathbb{Z}} \mathbb{C}.$$

For odd n this vector space is zero, while for even n it is naturally isomorphic to

$$HH_0(\text{End}_G(V_{x_0})) \cong S_{A^G}(x_0).$$

Since the map

$$\check{H}^0(X/G; S_{A^G}) \rightarrow \check{H}^0(\{x_0\}; S_{A^G}) = S_{A^G}(x_0)$$

is already surjective, it must be bijective, and $\check{H}^n(X/G; S_{A^G})$ must be zero for $n > 0$. \square

5 Some representation theory

We discuss the representation theory of \mathbb{H} and \mathbb{H}' , via parabolically induced representations and intertwining operators. Most results here rely on the author's previous work [Sol3].

As on page 10, let Γ be a finite group of diagram automorphisms of (R, Π) . Consider the group $W' = \Gamma \ltimes W$ and the algebra $\mathbb{H}' = \Gamma \ltimes \mathbb{H}$.

Proposition 5.1. a) *We have*

$$Z(\mathbb{H}') \supset S(\mathfrak{t}^*)^{W'} = \mathcal{O}(\mathfrak{t}/W'),$$

with equality if the action of W' on \mathfrak{t}^ is faithful.*

b) \mathbb{H}' is a finite type $\mathcal{O}(\mathfrak{t}/W')$ -algebra.

Proof. a) Lusztig [Lus2, Theorem 6.5] proved the corresponding statement for the graded Hecke algebra $\tilde{\mathbb{H}}(\tilde{\mathcal{R}})$ with formal parameters \mathbf{k}_α . We will adopt his argument to our setting.

On the one hand, it is clear from the multiplication rules (16) and (22) that $S(\mathfrak{t}^*)^{W'} \subset Z(\mathbb{H}')$. On the other hand, consider any

$$z = \sum_{w' \in W'} w' p_{w'} \in Z(\mathbb{H}'), \text{ with } p_{w'} \in S(\mathfrak{t}^*).$$

Assume that W' acts faithfully on \mathfrak{t}^* and that $p_{w'} \neq 0$ for some $w' \neq e$. Among these, pick one $w'_0 = \gamma_0 w_0 \in \Gamma \ltimes W$ such that w_0 is maximal with respect to the Bruhat order \leq for (W, Π) . For $x \in \mathfrak{t}^*$ a repeated application of (16) yields

$$\sum_{w' \in W'} w' p_{w'} x = zx = xz = \sum_{w' = \gamma w \in W'} \gamma \left(w w'(x) + \sum_{u \leq w} u f(\gamma(x), w, u) \right) p_{\gamma w},$$

for suitable polynomials $f(\gamma(x), w, u) \in S(\mathfrak{t}^*)$. Comparing the coefficients at w'_0 on both sides, we see that $w'_0(x) = x$. Since $x \in \mathfrak{t}^*$ was arbitrary and the action was faithful, this implies $w'_0 = e$, contrary to our assumption. Therefore $z \in Z(\mathbb{H}') \subset S(\mathfrak{t}^*)$. But now $w'z = zw'$ for all $w' \in W'$ shows that $z \in S(\mathfrak{t}^*)^{W'}$.

b) Clearly \mathbb{H}' is an $S(\mathfrak{t}^*)$ -module of finite rank, so it suffices to show that $S(\mathfrak{t}^*) = \mathcal{O}(\mathfrak{t})$ is of finite rank over $S(\mathfrak{t}^*)^{W'} = \mathcal{O}(\mathfrak{t}/W')$. For any $f \in S(\mathfrak{t}^*)$ the monic polynomial

$$P_f(X) := \prod_{w \in W'} (X - w(f))$$

has coefficients in $S(\mathfrak{t}^*)^{W'}$. Obviously $P_f(f) = 0$, so $S(\mathfrak{t}^*)$ is integral over $S(\mathfrak{t}^*)^{W'}$. Since $S(\mathfrak{t}^*)$ is finitely generated as an algebra, it follows that it is an $S(\mathfrak{t}^*)^{W'}$ -module of finite rank. \square

If an \mathbb{H}' -module V admits an $S(\mathfrak{t}^*)^{W'}$ -character $W'\lambda \in \mathfrak{t}/W'$, then we will refer to $W'\lambda$ as the central character of V . We say that a central character is real if it lies in \mathfrak{a}/W' .

A more subtle tool to study \mathbb{H}' -modules is restriction to the commutative subalgebra $S(\mathfrak{t}^*) \subset \mathbb{H}'$. Let (π, V) be an \mathbb{H}' -module and pick $\lambda \in \mathfrak{t}$. The λ -weight space of V is

$$V_\lambda = \{v \in V : \pi(x)v = \langle x, \lambda \rangle v \ \forall x \in \mathfrak{t}^*\}.$$

We call λ an $S(\mathfrak{t}^*)$ -weight of V if $V_\lambda \neq 0$.

Let $P \subset \Pi$ be a set of simple roots. They form a basis of a root subsystem $R_P \subset R$ with Weyl group $W_P \subset W$. Let $\mathfrak{a}_P \subset \mathfrak{a}$ and $\mathfrak{a}_P^* \subset \mathfrak{a}^*$ be the real spans of respectively R_P^\vee and R_P . We denote the complexifications of these vector spaces by \mathfrak{t}_P and \mathfrak{t}_P^* , and we write

$$\begin{aligned} \mathfrak{t}^P &= (\mathfrak{t}_P^*)^\perp = \{\lambda \in \mathfrak{t} : \langle x, \lambda \rangle = 0 \ \forall x \in \mathfrak{t}_P^*\}, \\ \mathfrak{t}^{P*} &= (\mathfrak{t}_P)^\perp = \{x \in \mathfrak{t}^* : \langle x, \lambda \rangle = 0 \ \forall \lambda \in \mathfrak{t}_P\}. \end{aligned}$$

We define the degenerate root data

$$\tilde{\mathcal{R}}_P = (\mathfrak{a}_P^*, R_P, \mathfrak{a}_P, R_P^\vee, P), \tag{36}$$

$$\tilde{\mathcal{R}}^P = (\mathfrak{a}^*, R_P, \mathfrak{a}, R_P^\vee, P), \tag{37}$$

and the graded Hecke algebras

$$\mathbb{H}_P = \mathbb{H}(\tilde{\mathcal{R}}_P, k), \quad (38)$$

$$\mathbb{H}^P = \mathbb{H}(\tilde{\mathcal{R}}^P, k). \quad (39)$$

Notice that the latter decomposes as a tensor product of algebras:

$$\mathbb{H}^P = S(\mathfrak{t}^{P*}) \otimes \mathbb{H}_P. \quad (40)$$

In particular every irreducible \mathbb{H}^P -module is of the form $\mathbb{C}_\lambda \otimes V$, where $\lambda \in \mathfrak{t}^P$ and V is an irreducible \mathbb{H}_P -module. In general, for any \mathbb{H}_P -module (ρ, V_ρ) and $\lambda \in \mathfrak{t}^P$ we denote the action of \mathbb{H}^P on $\mathbb{C}_\lambda \otimes V_\rho$ by ρ_λ . We define the parabolically induced module

$$\pi(P, \rho, \lambda) = \text{Ind}_{\mathbb{H}_P}^{\mathbb{H}}(\mathbb{C}_\lambda \otimes V_\rho) = \text{Ind}_{\mathbb{H}_P}^{\mathbb{H}}(\rho_\lambda) = \mathbb{H} \otimes_{\mathbb{H}^P} V_{\rho_\lambda}. \quad (41)$$

Since the complex vector space \mathfrak{t} has a distinguished real form \mathfrak{a} , we can decompose any $\lambda \in \mathfrak{t}$ unambiguously as

$$\lambda = \Re(\lambda) + i\Im(\lambda) \text{ with } \Re(\lambda), \Im(\lambda) \in \mathfrak{a}. \quad (42)$$

We define the positive cones

$$\begin{aligned} \mathfrak{a}^{*+} &= \{x \in \mathfrak{a}^* : \langle x, \alpha^\vee \rangle \geq 0 \forall \alpha \in \Pi\}, \\ \mathfrak{a}_P^+ &= \{\mu \in \mathfrak{a}_P : \langle \alpha, \mu \rangle \geq 0 \forall \alpha \in P\}, \\ \mathfrak{a}^{P+} &= \{\mu \in \mathfrak{a}^P : \langle \alpha, \mu \rangle \geq 0 \forall \alpha \in \Pi \setminus P\}, \\ \mathfrak{a}^{P++} &= \{\mu \in \mathfrak{a}^P : \langle \alpha, \mu \rangle > 0 \forall \alpha \in \Pi \setminus P\}. \end{aligned} \quad (43)$$

The antidual of \mathfrak{a}^{*+} is

$$\mathfrak{a}^- = \{\lambda \in \mathfrak{a} : \langle x, \lambda \rangle \leq 0 \forall x \in \mathfrak{a}^{*+}\} = \{\sum_{\alpha \in \Pi} \lambda_\alpha \alpha^\vee : \lambda_\alpha \leq 0\}. \quad (44)$$

The interior \mathfrak{a}^{--} of \mathfrak{a}^- equals

$$\{\sum_{\alpha \in \Pi} \lambda_\alpha \alpha^\vee : \lambda_\alpha < 0\}$$

if Π spans \mathfrak{a}^* , and is empty otherwise. A finite dimensional \mathbb{H} -module V is called tempered if $\Re(\lambda) \in \mathfrak{a}^-$, for all weights λ . More restrictively we say that V belongs to the discrete series if it is irreducible and $\Re(\lambda) \in \mathfrak{a}^{--}$, again for all weights λ .

For now on we assume that the root system R is crystallographic and that k is real, that is, $k_\alpha \in \mathbb{R}$ for all $\alpha \in \Pi$. Of particular importance are the parabolically induced representations $\pi(P, \delta, \lambda)$ where (δ, V_δ) is a discrete series representation of \mathbb{H}_P . We call such a triple (P, δ, λ) an induction datum and we denote the space of these by $\tilde{\Xi}$. For $\xi, \eta \in \tilde{\Xi}$ we write $\xi \cong \eta$ if $\xi = (P, \delta, \lambda)$ and $\eta = (P, \sigma, \lambda)$ with $\delta \cong \sigma$ as \mathbb{H}_P -modules. The space of unitary induction data is

$$\tilde{\Xi}_u := \{(P, \delta, \lambda) \in \tilde{\Xi} : \lambda \in i\mathfrak{a}^P\}.$$

There exists a natural involution on \mathbb{H} such that $\pi(\xi)$ is a unitary module if and only if $\xi \in \tilde{\Xi}_u$, see [Sol3, Section 6]. Moreover, by [Sol3, Proposition 7.2] every irreducible \mathbb{H} -module is a quotient of some $\pi(P, \delta, \lambda)$ with $\lambda \in \mathfrak{a}^{P+}$. Thus discrete series representations can be considered as the basic objects for constructing \mathbb{H} -modules, and it is certainly useful to know more about them.

Theorem 5.2. *There are only finitely many equivalence classes of discrete series representations, and the $S(\mathfrak{t}^*)$ -weights of a discrete series representation all lie in \mathfrak{a} .*

Proof. See Lemma 2.13 and Corollary 2.14 of [Slo], or [Sol3, Theorem 6.2]. We note that these proofs depend on the classification of possible $S(\mathfrak{t}^*)$ -weights in [HeOp, Section 4]. \square

The strategy of parabolic induction can be extended from \mathbb{H} to $\mathbb{H}' = \Gamma \ltimes \mathbb{H}$. For an induction datum $(P, \delta, \lambda) \in \Xi$ we put

$$\pi'(P, \delta, \lambda) = \text{Ind}_{\mathbb{H}}^{\mathbb{H}'} \pi(P, \delta, \lambda) = \text{Ind}_{\mathbb{H}^P}^{\mathbb{H}'} (\delta_\lambda).$$

For $P, Q \subset \Pi$ we write

$$W'(P, Q) = \{w \in W' : w(P) = Q\}.$$

Any $w \in W'(P, Q)$ induces algebra isomorphisms

$$\begin{aligned} \psi_w : \mathbb{H}_P &\rightarrow \mathbb{H}_Q \\ \psi_w : \mathbb{H}^P &\rightarrow \mathbb{H}^Q \\ \psi_w(xu) &= w(x) w u w^{-1} \quad x \in \mathfrak{t}^*, u \in W_P. \end{aligned} \tag{45}$$

We agree to call a $\Gamma \ltimes \mathbb{H}$ -module tempered if its restriction to \mathbb{H} is tempered. Let (π, V) be any \mathbb{H} -module and let $\gamma \in \Gamma$. Because $\Gamma \cdot \mathfrak{a}^- = \mathfrak{a}^-$, the \mathbb{H} -module $(\pi \circ \psi_\gamma^{-1}, V)$ is tempered, respectively discrete series, if and only if π is tempered, respectively discrete series.

For $w \in W'(P, Q)$ and $(P, \delta, \lambda) \in \tilde{\Xi}$ we have

$$w(P, \delta, \lambda) := (Q, \delta \circ \psi_w^{-1}, w(\lambda)) \in \tilde{\Xi}.$$

This defines a partial action of W' on $\tilde{\Xi}$, which preserves $\tilde{\Xi}_u$. For $\xi \in \tilde{\Xi}$ we write

$$W'_\xi = \{w \in W' : w(\xi) \cong \xi\}.$$

Induction data that are W' -associate usually, but not always, yield equivalent \mathbb{H}' -modules:

Proposition 5.3. *Let w, P, δ and λ be as above. There exists an intertwining operator*

$$\pi'(w, P, \delta, \lambda) : \pi'(P, \delta, \lambda) \rightarrow \pi'(w(P, \delta, \lambda)),$$

which is rational as a function of $\lambda \in \mathfrak{t}^P$. It is regular and invertible for λ in a nonempty Zariski-open subset of \mathfrak{t}^P .

Proof. See Proposition 3.3 and page 28 of [Sol3]. \square

Recall that R was crystallographic and that $k_\alpha \in \mathbb{R}$ for all $\alpha \in \Pi$. For unitary induction data much more can be said:

Theorem 5.4. *Let $\xi, \eta \in \tilde{\Xi}_u$.*

a) The \mathbb{H}' -module $\pi'(\xi)$ is tempered and completely reducible.

b) The operators

$$\{\pi'(w, \xi) : w \in W', w(\xi) \cong \eta\}$$

are well-defined and invertible.

c) These operators span $\text{Hom}_{\mathbb{H}'}(\pi'(\xi), \pi'(\eta))$.

d) $\{\pi'(\xi)(h) : h \in \mathbb{H}'\} = \{M \in \text{End}_{\mathbb{C}}(\pi'(\xi)) : \pi'(w, \xi)M\pi'(w, \xi)^{-1} = M \forall w \in W'_\xi\}$.

Proof. a) is the combination of Corollary 6.4 and Theorem A.1.c of [Sol3].

b) and c) are in [Sol3, Theorem 8.2.c].

d) By parts b) and c) the right-hand side is the bicommutant of the left-hand side. By part a) $\pi'(\xi)(\mathbb{H}')$ is a semisimple algebra, hence equal to its own bicommutant in $\text{End}_{\mathbb{C}}(\pi'(\xi))$. \square

6 The primitive ideal spectrum

We will describe the primitive ideal spectrum of $\mathbb{H}' = \Gamma \rtimes \mathbb{H}$ as explicitly as possible. First we stratify it and we describe the strata as topological spaces. After that we will determine the cohomology of the sheaf corresponding to $\text{Prim}(\mathbb{H}')$ and draw some conclusions about the periodic cyclic homology of \mathbb{H}' . Throughout this section we assume that the root system R is crystallographic and that $k_\alpha \in \mathbb{R}$ for all $\alpha \in \Pi$.

We denote the central character of an (irreducible) \mathbb{H}_P -representation by $cc_P(\delta) \in \mathfrak{t}_P/W_P$. Since W_P acts orthogonally on \mathfrak{t} , all $S(\mathfrak{t}_P^*)$ -weights of δ have the same norm, which allows us to write $\|cc_P(\delta)\|$ without ambiguity.

Theorem 6.1. [Sol3, Theorem 8.3.b]

Let ρ be an irreducible \mathbb{H}' -module. There exists a unique W' -association class $W'\xi_\rho = W'(P_\rho, \delta_\rho, \lambda_\rho)$ in $\tilde{\Xi}/W'$ such that:

- ρ is a constituent of $\pi'(\xi_\rho)$,
- $\|cc_{P_\rho}(\delta_\rho)\|$ is maximal among induction data with the first property.

Lemma 6.2. Let ρ be an irreducible \mathbb{H}' -module with $W'\xi_\rho = W'(P_\rho, \delta_\rho, \lambda_\rho)$.

a) ρ is tempered if and only if $\xi_\rho \in \tilde{\Xi}_u$, which is equivalent to $\lambda_\rho \in i\mathfrak{a}$.

b) The central character of ρ lies in \mathfrak{a}/W' if and only if $\lambda_\rho \in \mathfrak{a}$.

Proof. a) follows from Proposition 7.3.c and Theorem 8.3.a of [Sol3].

b) The central character of $\pi'(P, \delta, \lambda)$ is $W'(cc_P(\delta) + \lambda)$ and by Theorem 5.2 $cc_P(\delta) \in \mathfrak{a}/W_P$. \square

For every $P \subset \Pi$, every discrete series representation δ of \mathbb{H}_P and every $U \subset \mathfrak{t}^P$ we get a subset of $\text{Prim}(\mathbb{H}')$:

$$\text{Prim}_{(P, \delta, U)}(\mathbb{H}') := \{\ker \rho \in \text{Prim}(\mathbb{H}') : W'\xi_\rho \cap (P, \delta, U) \neq \emptyset\}.$$

For $U = \mathfrak{t}^P$ or $U = \{\lambda\}$ we abbreviate this to $\text{Prim}_{P,\delta}(\mathbb{H}')$ or $\text{Prim}_{(P,\delta,\lambda)}(\mathbb{H}')$.

The $\text{Prim}_{P,\delta}(\mathbb{H}')$ form a partition of $\text{Prim}(\mathbb{H}')$, but they are not closed. However, the boundary of $\text{Prim}_{P,\delta}(\mathbb{H}')$ can only contain elements of $\text{Prim}_{Q,\sigma}(\mathbb{H}')$ if $\|cc_Q(\sigma)\| > \|cc_P(\delta)\|$. By [Sol3, Theorem 6.2.b] there are only finitely many W' -association classes of pairs (P, δ) . Pick one (P_i, δ_i) from every W' -association class, and order them such that

$$\|cc_{P_i}(\delta_i)\| \leq \|cc_{P_j}(\delta_j)\| \text{ if } j \leq i.$$

Then $\bigcup_{j \leq i} \text{Prim}_{P_j, \delta_j}(\mathbb{H}')$ is closed in $\text{Prim}(\mathbb{H}')$ for every i . This defines a stratification of $\text{Prim}(\mathbb{H}')$, with strata $\text{Prim}_{P,\delta}(\mathbb{H}')$. This stratification is the analogue of a stratification of the smooth dual of a reductive p -adic group [Sol2, Lemma 2.17], which for GL_n can already be found in [ScZi, Section 9].

The group

$$W'_\delta := \{w \in W' : w(P) = P, \delta \circ \psi_w^{-1} \cong \delta\}$$

acts on \mathfrak{t}^P and λ 's in the same W'_δ -orbit give rise to the same elements of $\text{Prim}_{P,\delta}(\mathbb{H}')$. Hence there is a canonical map

$$\text{Prim}_{P,\delta}(\mathbb{H}') \rightarrow \mathfrak{t}^P / W'_\delta,$$

which is continuous, surjective and finite-to-one.

Proposition 6.3. [Sol3, Proposition 10.1]

Suppose that $U \subset \mathfrak{t}^P$ satisfies

- $U \rightarrow \{-1, 0, 1\} : \lambda \mapsto \text{sign}(\Re(\lambda), \alpha)$ is constant for all $\alpha \in \Pi \setminus P$,
- every $\lambda \in U$ has the same stabilizer in W'_δ .

Then $\text{Prim}_{(P,\delta,U)}(\mathbb{H}')$ is homeomorphic to $U/W'_\delta \times \text{Prim}_{(P,\delta,\lambda_0)}(\mathbb{H}')$, for any $\lambda_0 \in U$.

Thus $\text{Prim}_{P,\delta}(\mathbb{H}')$ looks like the quotient of the vector space \mathfrak{t}^P by W'_δ , but with certain linear subspaces carrying a multiplicity.

Consider the sheaf $S_{\mathbb{H}'}$ over \mathfrak{t}/W' , as in Definition 4.4. It has a subsheaf $S_{P,\delta}$ consisting of all those sections of $S_{\mathbb{H}'}$ that use only elements of $\text{Prim}_{P,\delta}(\mathbb{H}')$. Alternatively we can obtain $S_{P,\delta}$ as the sheaf corresponding to a certain subquotient algebra of \mathbb{H}' , but there is no need to use this subquotient.

Lemma 6.4.

$$\check{H}^n(\mathfrak{t}/W'; S_{P,\delta}) = 0 \quad \forall n > 0,$$

and evaluation at $\mu = W'cc_P(\delta) \in \mathfrak{t}/W'$ induces a linear bijection

$$\check{H}^0(\mathfrak{t}/W'; S_{P,\delta}) \longrightarrow S_{P,\delta}(\mu) = \mathbb{C}\{I \in \text{Prim}(\mathbb{H}') : I \supset \ker(\pi'(P, \delta, 0))\}.$$

Proof. By construction $S_{P,\delta}$ is the direct image of a sheaf \mathcal{F}_δ on \mathfrak{t}^P/W'_δ , with respect to the map

$$\mathfrak{t}^P/W'_\delta \rightarrow \mathfrak{t}^P : W'_\delta \lambda \mapsto W'(\lambda + cc_P(\delta)). \quad (46)$$

According to [Bre, Theorem 11.1] this induces a natural isomorphism

$$\check{H}^*(\mathfrak{t}/W'; S_{P,\delta}) \cong \check{H}^*(\mathfrak{t}^P/W'_\delta; \mathcal{F}_\delta). \quad (47)$$

By Proposition 5.3 and Theorem 5.4.b we can find a ball $B_r \subset \mathfrak{t}^P$ around 0, of radius $r > 0$, such that all the operators $\pi'(w, P, \delta, \lambda)$ with $w \in W'_\delta$ and $\lambda \in B_r$ are regular and invertible. By Proposition 6.3 the homeomorphism

$$B_r \rightarrow \mathfrak{t}^P : \lambda \mapsto \tan\left(\frac{\pi\|\lambda\|}{2r}\right)\lambda$$

induces a sheaf isomorphism $\mathcal{F}_\delta|_{B_r/W'_\delta} \rightarrow \mathcal{F}_\delta$. Hence

$$\check{H}^*(\mathfrak{t}/W'; S_{P,\delta}) \cong \check{H}^*(B_r/W'_\delta; \mathcal{F}_\delta). \quad (48)$$

Let $V = \mathbb{H}' \otimes_{\mathbb{H}^P} V_\delta$ be the vector space on which all the representations $\pi'(P, \delta, \lambda)$ are realized, and consider the Fréchet algebra $A = C(B_r) \otimes \text{End}_{\mathbb{C}}(V)$. For $w \in W'_\delta$ we define

$$u_w \in C(B_r) \otimes \text{Aut}_{\mathbb{C}}(V) \quad \text{by} \quad u_w(\lambda) = \pi'(w, P, \delta, w^{-1}\lambda).$$

Now W'_δ acts on A by

$$(w \cdot a)(\lambda) = \pi'(w, P, \delta, w^{-1}\lambda)a(w^{-1}\lambda)\pi'(w, P, \delta, w^{-1}\lambda)^{-1},$$

which can be abbreviated to $w \cdot a = u_w(a \circ w^{-1})u_w^{-1}$. Thus we are in the setting of page 20. We will relate our sheaves to the algebra $A' = A^{W'_\delta}$ of W'_δ -invariants. By Theorem 5.4.d we have

$$\{\pi'(P, \delta, \lambda)(h) : h \in \mathbb{H}'\} = \{a(\lambda) : a \in A'\}$$

for all $\lambda \in i\mathfrak{a}^P$. Hence the sheaves F_δ and $S_{A'}$ have the same restriction to $(B_r \cap i\mathfrak{a}^P)/W'_\delta$. For every $\lambda \in B_r$ the dimension of $S_{A'}(W'_\delta\lambda)$ equals the number of inequivalent constituents of the projective $(W'_\delta)_\lambda$ -representation $w \mapsto u_w(\lambda)$. Since $(\mathfrak{t}^P)^G$ is connected for every $G \subset W'_\delta$, this number depends only on λ via the isotropy group $(W'_\delta)_\lambda$. Combining this with Proposition 6.3, we see that the sheaves $\mathcal{F}_\delta|_{B_r/W'_\delta}$ and $S_{A'}$ are isomorphic. According to Lemma 4.6

$$\check{H}^n(B_r/W'_\delta; S_{A'}) = 0 \quad \forall n > 0$$

and the evaluation map

$$\check{H}^0(B_r/W'_\delta; S_{A'}) \rightarrow S_{A'}(0)$$

is bijection. Consequently

$$\check{H}^n(\mathfrak{t}/W'; S_{P,\delta}) = \check{H}^n(B_r/W'_\delta; \mathcal{F}_\delta) \quad \forall n > 0$$

and the maps

$$\check{H}^0(\mathfrak{t}^P/W'; \mathcal{F}_\delta) = \check{H}^0(B_r/W'_\delta; \mathcal{F}_\delta) \rightarrow \mathcal{F}_\delta(0)$$

are isomorphisms. In view of (46) and (47) this shows that the corresponding evaluation

$$\check{H}^0(\mathfrak{t}/W'; S_{P,\delta}) \rightarrow S_{P,\delta}(W'_{ccP}(\delta))$$

is also bijective. \square

We denote the finite set of equivalence classes of irreducible tempered \mathbb{H}' -modules with real central character by $\text{Irr}_0(\mathbb{H}')$. In every such equivalence class we pick one \mathbb{H}' -module (π, V) and we confuse $\text{Irr}_0(\mathbb{H}')$ with this set of modules. Let $\mathbb{C}\text{Irr}_0(\mathbb{H}')$ be the complex vector space with basis $\text{Irr}_0(\mathbb{H}')$.

By Theorem 5.4.a and Lemma 6.2 $\text{Irr}_0(\mathbb{H}')$ consists precisely of the irreducible summands of the modules $\pi'(P, \delta, 0)$, where $P \subset \Pi$ and δ is a discrete series representation of \mathbb{H}_P . Notice that all the primitive ideals appearing in Lemma 6.4 correspond to such modules.

Theorem 6.5. *As before we assume that R is crystallographic and that k is real.*

a) $\check{H}^n(\mathfrak{t}/W'; S_{\mathbb{H}'}) = 0$ for all $n > 0$, and there exists a canonical linear bijection

$$\check{H}^0(\mathfrak{t}/W'; S_{\mathbb{H}'}) \rightarrow \mathbb{C}\text{Irr}_0(\mathbb{H}').$$

b) *The natural algebra homomorphisms*

$$\mathbb{C}[W'] \longrightarrow \mathbb{H}' \longrightarrow \bigoplus_{(\pi, V) \in \text{Irr}_0(\mathbb{H}')} \text{End}_{\mathbb{C}}(V)$$

induce isomorphisms on periodic cyclic homology.

c) *The W' -representations $\{V|_{W'} : (\pi, V) \in \text{Irr}_0(\mathbb{H}')\}$ form a \mathbb{Q} -basis of the representation ring $R(W') \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. a) On page 25 we defined a stratification of $\text{Prim}(\mathbb{H}')$ with terms

$$F_i := \bigcup_{j \leq i} \text{Prim}_{P_j, \delta_j}(\mathbb{H}').$$

These give rise to subsheaves S_i of $S_{\mathbb{H}'}$, consisting of all sections of $S_{\mathbb{H}'}$ that use only elements of F_i . By construction S_i/S_{i-1} is isomorphic to the sheaf S_{P_i, δ_i} studied in Lemma 6.4. This filtration of $S_{\mathbb{H}'}$ yields a spectral sequence that converges to $\check{H}^*(\mathfrak{t}/W'; S_{\mathbb{H}'})$ and has initial terms $E_{i,j}^1 = \check{H}^{i+j}(\mathfrak{t}/W'; S_{P_i, \delta_i})$. By Lemma 6.4 $E_{i,j}^1 = 0$ unless $i + j = 0$. Hence all the boundary maps $\delta_{i,j}^r : E_{i,j}^r \rightarrow E_{i-r, j+r-1}^r$ are 0 for $r \geq 1$, and the spectral sequence degenerates at $E_{*,*}^1$. Lemma 6.4 tells us that

$$\begin{aligned} \check{H}^0(\mathfrak{t}/W'; S_{\mathbb{H}'}) &\cong \bigoplus_i \check{H}^0(\mathfrak{t}/W'; S_{P_i, \delta_i}) \\ &\cong \bigoplus_i \mathbb{C}\{I \in \text{Prim}(\mathbb{H}') : I \supset \ker(\pi'(P_i, \delta_i, 0))\} \cong \mathbb{C}\text{Irr}_0(\mathbb{H}'), \end{aligned}$$

where the composite isomorphism is given by evaluating a global section of $S_{\mathbb{H}'}$ at all points corresponding to $\text{Irr}_0(\mathbb{H}')$.

b) follows from a) and Theorems 4.5 and 3.3.

c) Since $\mathbb{C}[W']$ is finite dimensional and semisimple,

$$HP_*(\mathbb{C}[W']) \cong HH_0(\mathbb{C}[W']) = \mathbb{C}[W']/[\mathbb{C}[W'], \mathbb{C}[W']] \cong Z(\mathbb{C}[W']) \cong R(W') \otimes_{\mathbb{Z}} \mathbb{C}.$$

Under these isomorphisms an irreducible W' -representation ρ corresponds to the class of the central idempotent $e_\rho \in Z(\mathbb{C}[W'])$ in $HH_0(\mathbb{C}[W'])$. Similarly

$$\begin{aligned} HP_*\left(\bigoplus_{(\pi, V) \in \text{Irr}_0(\mathbb{H}')} \text{End}_{\mathbb{C}}(V)\right) &= \bigoplus_{(\pi, V) \in \text{Irr}_0(\mathbb{H}')} HH_0(\text{End}_{\mathbb{C}}(V)) \\ &\cong \bigoplus_{(\pi, V) \in \text{Irr}_0(\mathbb{H}')} Z(\text{End}_{\mathbb{C}}(V)) \cong \mathbb{C}\text{Irr}_0(\mathbb{H}'). \end{aligned}$$

Let $HP_*(\phi)$ be the map induced by the algebra homomorphism

$$\phi : \mathbb{C}[W'] \rightarrow \bigoplus_{(\pi, V) \in \text{Irr}_0(\mathbb{H}')} \text{End}_{\mathbb{C}}(V).$$

By construction

$$HP_*(\phi)(e_\rho) = \sum_{(\pi, V) \in \text{Irr}_0(\mathbb{H}')} \text{tr}(\pi(e_\rho)) (\pi, V)$$

By part b) this is a bijection

$$R(W') \otimes_{\mathbb{Z}} \mathbb{C} \cong Z(\mathbb{C}[W']) \rightarrow \mathbb{C} \text{Irr}_0(\mathbb{H}').$$

We declare the canonical bases $\widehat{W'}$ and $\text{Irr}_0(\mathbb{H}')$ of these vector spaces to be orthonormal. Then the adjoint map

$$HP_*(\phi)^* : \mathbb{C} \text{Irr}_0(\mathbb{H}') \rightarrow R(W') \otimes_{\mathbb{Z}} \mathbb{C}$$

is also bijective, and it sends (π, V) to $\pi \circ \phi \in R(W')$. Hence

$$\{\pi \circ \phi : (\pi, V) \in \text{Irr}_0(\mathbb{H}')\}$$

is a \mathbb{C} -basis of $R(W') \otimes_{\mathbb{Z}} \mathbb{C}$ and a \mathbb{Q} -basis of $R(W') \otimes_{\mathbb{Z}} \mathbb{Q}$. \square

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